A functional central limit theorem for interacting particle systems on transitive graphs

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Abstract

A finite range interacting particle system on a transitive graph is considered. Assuming that the dynamics and the initial measure are invariant, the normalized empirical distribution process converges in distribution to a centered diffusion process. As an application, a central limit theorem for certain hitting times, interpreted as failure times of a coherent system in reliability, is derived.

Key words : Interacting particle system, functional central limit theorem, hitting time.

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1 Introduction

Interacting particle systems have attracted a lot of attention because of their versatile modelling power (see for instance [?, ?]). However, most available results deal with their asymptotic behavior, and relatively few theorems describe their transient regime. In particular, central limit theorems for random fields have been available for a long time [?, ?, ?, ?, ?, ?, ?], diffusion approximations and invariance principles have an even longer history ([?] and references therein), but those functional central limit theorems that describe the transient behavior of an interacting particle system are usually much less general than their fixed-time counterparts. Existing results (see [?, ?, ?, ?]) require rather stringent hypotheses: spin flip dynamics on $\mathbb{Z}$, reversibility, exponential ergodicity, stationarity... (see Holley and Strook's discussion in the introduction of [?]). The main objective of this article is to prove a functional central limit theorem for interacting particle systems, under very mild hypotheses, using some new techniques of weakly dependent random fields.

Our basic reference on interacting particle systems is the textbook by Liggett [?], and we shall try to keep our notations as close to his as possible: S denotes the (countable) set of sites, W the (finite) set of states, $\mathcal{X} = W^S$ the set of configurations, and $\{\eta_t\,,\ t \geq 0\}$ an interacting particle system, i.e. a Feller process with values in $\mathcal{X}$. If R is a finite subset of S, an empirical process is defined by counting how many sites of R are in each possible state at time t. This empirical process will be denoted by $N^R = \{N_t^R\,,\ t \geq 0\}$, and defined as follows.

$$N_t^R = (N_t^R(w))_{w \in W}\ , \quad N_t^R(w) = \sum_{x \in R} \mathbb{I}_w(\eta_t(x))\ ,$$

where $\mathbb{I}_w$ denotes the indicator function of state w. Thus N_t^R is a $\mathbb{N}^W$-valued stochastic process, which is not Markovian in general. Our goal is to show that, under suitable hypotheses, a properly scaled version of N^R converges to a Gaussian process as R increases to S. The hypotheses will be precised in sections 2 and 3 and the main result (Theorem 4.1) will be stated and proved in section 4. Here is a loose description of our assumptions. Dealing with a sum of random variables, two hypotheses can be made for a central limit theorem: weak dependence and identical distributions.

1. *Weak dependence:* In order to give it a sense, one has to define a distance between sites, and therefore a graph structure. We shall first suppose that this (undirected) graph structure has bounded degree. We shall assume also finite range interactions: the configuration can simultaneously change only on a bounded set of sites, and its value at one site can influence transition rates only up to a fixed distance (Definition 3.2). Then if f and g are two functions whose dependence on the coordinates decreases exponentially fast with the distance from two distant finite sets R_1 and R_2, we shall prove that the covariance between $f(\eta_s)$ and $g(\zeta_t)$ decays exponentially fast in the distance between R_1 and R_2 (Proposition 3.3). The central limit theorem 4.1 will actually be proved in a much narrower setting, that of group invariant dynamics on a transitive graph (Definition 3.4). However we believe that a covariance inequality for general finite range interacting particle systems is of independent interest. Of course

the bound of Proposition 3.3 is not uniform in time, without further assumptions.

2. *Identical distributions:* In order to ensure that the indicator processes $\{\mathbb{I}_w(\eta_t(x))\,,\ t \geq 0\}$ are identically distributed, we shall assume that the set of sites S is endowed with a transitive graph structure (see [?] as a general reference), and that both the transition rates and the initial distribution are invariant by the automorphism group action. This generalizes the notion of translation invariance, usually considered in $\mathbb{Z}^d$ ([?] p. 36), and can be applied to non-lattice graphs such as trees. Several recent articles have shown the interest of studying random processes on graph structures more general than $\mathbb{Z}^d$ lattices: see e.g. [?, ?, ?], and for general references [?, ?].

Among the potential applications of our result, we chose to focus on the hitting time of a prescribed level by a linear combination of the empirical process. In [?], such hitting times were considered in the application context of reliability. Indeed the sites in R can be viewed as components of a coherent system and their states as degradation levels. Then a linear combination of the empirical process is interpreted as the global degradation of the system, and by Theorem 4.1, it is asymptotically distributed as a diffusion process if the number of components is large. An upper bound for the degradation level can be prescribed: the system is working as soon as the degradation is lower, and fails at the hitting time. More precisely, let $f\ : w \mapsto f(w)$ be a mapping from W to $\mathbb{R}$. The total degradation is the real-valued process $D^R = \{D^R_t\,,\ t \geq 0\}$, defined by:

$$D^R_t = \sum_{w \in W} f(w) N^R_t(w).$$

If a is the prescribed level, the failure time of the system will be defined as the random variable

$$T^R_a = \inf\{t\ \geq 0\,,\ D^R_t \geq a\,\}.$$

Under suitable hypotheses, we shall prove that T^R_a converges weakly to a normal distribution, thus extending Theorem 1.1 of [?] to systems with dependent components. In reliability (see [?] for a general reference), components of a coherent system are usually considered as independent. The reason seems to be mathematical convenience rather than realistic modelling. Models with dependent components have been proposed in the setting of stochastic Petri nets [?, ?]. Observing that a Markovian Petri net can also be interpreted as an interacting particle system, we believe that the model studied here is versatile enough to be used in practical applications.

The paper is organized as follows. Some basic facts about interacting particle systems are first recalled in section 2. They are essentially those of sections I.3 and I.4 of [?], summarized here for sake of completeness, and in order to fix notations. The covariance inequality for finite range interactions and local functions will be given in section 3. Our main result, Theorem 4.1, will be stated in section 4. Some examples of transitive graphs are proposed in section 5. The application to hitting times and their reliability interpretation is the object of section 6. In the proof of Theorem 4.1, we need a spatial CLT for an interacting particle system at fixed time, i.e. a random field. We thought interesting to

state it independently in section 7: Proposition 7.1 is in the same vein as the one proved by Bolthausen [?] on $\mathbb{Z}^d$, but it uses a somewhat different technique. All proofs are postponed to section 8.

2 Main notations and assumptions

In order to fix notations, we briefly recall the basic construction of general interacting particle systems, described in sections I.3 and I.4 of Liggett's book [?].

Let S be a countable set of sites, W a finite set of states, and $\mathcal{X} = W^S$ the set of configurations, endowed with its product topology, that makes it a compact set. One defines a Feller process on $\mathcal{X}$ by specifying the local transition rates: to a configuration η and a finite set of sites T is associated a nonnegative measure $c_T(\eta, \cdot)$ on W^T. Loosely speaking, we want the configuration to change on T after an exponential time with parameter

$$c_{T,\eta} = \sum_{\zeta \in W^T} c_T(\eta, \zeta).$$

After that time, the configuration becomes equal to ζ on T, with probability $c_T(\eta,\zeta)/c_{T,\eta}$. Let η^ζ denote the new configuration, which is equal to ζ on T, and to η outside T. The infinitesimal generator should be:

$$\Omega f(\eta) = \sum_{T \subset S} \sum_{\zeta \in W^T} c_T(\eta,\zeta)(f(\eta^\zeta) - f(\eta)). \tag{1}$$

For Ω to generate a Feller semigroup acting on continuous functions from X into $\mathbb{R}$, some hypotheses have to be imposed on the transition rates $c_T(\eta, \cdot)$.

The first condition is that the mapping $\eta \mapsto c_T(\eta, \cdot)$ should be continuous (and thus bounded, since $\mathcal{X}$ is compact). Let us denote by c_T its supremum norm.

$$c_T = \sup_{\eta \in X} c_{T,\eta}.$$

It is the maximal rate of change of a configuration on T. One essential hypothesis is that the maximal rate of change of a configuration at one given site is bounded.

$$B = \sup_{x \in S} \sum_{T \ni x} c_T < \infty. \tag{2}$$

If f is a continuous function on $\mathcal{X}$, one defines $\Delta_f(x)$ as the degree of dependence of f on x:

$$\Delta_f(x) = \sup\{\, |f(\eta) - f(\zeta)|\,,\ \eta, \zeta \in X \text{ and } \eta(y) = \zeta(y)\ \forall\, y \neq x\,\}.$$

Since f is continuous, $\Delta_f(x)$ tends to 0 as x tends to infinity, and f is said to be *smooth* if Δ_f is summable:

$$|||f||| = \sum_{x \in S} \Delta_f(x) < \infty.$$

It can be proved that if f is smooth, then Ωf defined by (1) is indeed a continuous function on $\mathcal{X}$ and moreover:

$$\|\Omega f\| \leq B |\!|\!| f |\!|\!|.$$

We also need to control the dependence of the transition rates on the configuration at other sites. If $y \in S$ is a site, and $T \subset S$ is a finite set of sites, one defines

$$c_T(y) = \sup\{ \, \|c_T(\eta_1, \cdot\,) - c_T(\eta_2, \cdot\,)\|_{tv} \, , \; \eta_1(z) = \eta_2(z) \; \forall \, z \neq y \, \},$$

where $\| \cdot \|_{tv}$ is the total variation norm:

$$\|c_T(\eta_1, \cdot\,) - c_T(\eta_2, \cdot\,)\|_{tv} = \frac{1}{2} \sum_{\zeta \in W^T} |c_T(\eta_1, \zeta) - c_T(\eta_2, \zeta)|.$$

If x and y are two sites such that $x \neq y$, the *influence* of y on x is defined as:

$$\gamma(x, y) = \sum_{T \ni x} c_T(y).$$

We will set $\gamma(x, x) = 0$ for all x. The influences $\gamma(x, y)$ are assumed to be summable:

$$M = \sup_{x \in S} \sum_{y \in S} \gamma(x, y) < \infty. \tag{3}$$

Under both hypotheses (2) and (3), it can be proved that the closure of Ω generates a Feller semigroup $\{S_t \, , \, t \geq 0\}$ (Theorem 3.9 p. 27 of [**?**]). A generic process with semigroup $\{S_t \, , \, t \geq 0\}$ will be denoted by $\{\eta_t \, , \, t \geq 0\}$. Expectations relative to its distribution, starting from $\eta_0 = \eta$ will be denoted by $\mathbb{E}_\eta$. For each continuous function f, one has:

$$S_t f(\eta) = \mathbb{E}_\eta[f(\eta_t)] = \mathbb{E}[f(\eta_t) \, | \, \eta_0 = \eta].$$

Assume now that W is ordered, (say $W = \{1, \ldots, n\}$). Let $\mathcal{M}$ denote the class of all continuous functions on X which are monotone in the sense that $f(\eta) \leq f(\xi)$ whenever $\eta \leq \xi$. As it was noticed by Liggett (1985) it is essential to take advantage of monotonicity in order to prove limit theorems for particle systems. The following theorems discuss a number of ideas related to monotonicity.

Theorem 2.1 (Theorem 2.2 Liggett, (1985)) *Suppose η_t is a Feller process on X with semigroup $S(t)$. The following statement are equivalent :*

(a) $f \in \mathcal{M}$ implies $S(t)f \in \mathcal{M}$, for all $t \geq 0$

(b) $\mu_1 \leq \mu_2$ implies $\mu_1 S(t) \leq \mu_2 S(t)$ for all $t \geq 0$.

Recall that $\mu_1 \leq \mu_2$ provided that $\int f d\mu_1 \leq \int f d\mu_2$ for any $f \in \mathcal{M}$.

Definition 2.2 *A Feller process is said to be monotone (or attractive) if the equivalent conditions of Theorem 2.1 are satisfied.*

Theorem 2.3 (Theorem 2.14 Liggett, (1985)) *Suppose that $S(t)$ and Ω are respectively the semigroup and the generator of a* **monotone** *Feller process on X. Assume further that Ω is a* **bounded** *operator. Then the following two statements are equivalent:*

(a) $\Omega fg \geq f\Omega g + g\Omega f$, for all $f,\ g \in \mathcal{M}$

(b) $\mu S(t)$ has positive correlations whenever μ does.

Recall that μ has positive correlation if $\int fgd\mu \geq \left(\int fd\mu\right)\left(\int gd\mu\right)$ for any $f, g \in \mathcal{M}$.

The following corollary gives conditions under which the positive correlation property continue to hold at later times if it holds initially.

Corollary 2.4 *[Corollary 2.21 Liggett, (1985)] Suppose that the assumptions of Theorem 2.3 are satisfied and that the equivalent conditions of Theorem 2.3 hold. Let η_t be the corresponding process, where the distribution of η_0 has positive correlations. Then for $t_1 < t_2 < \cdots < t_n$ the joint distribution of $(\eta_{t_1}, \cdots, \eta_{t_n})$, which is a probability measure on X^n, has positive correlations.*

3 Covariance inequality

This section is devoted to the covariance of $f(\eta_s)$ and $g(\eta_t)$ for a finite range interacting particle system when the underlying graph structure has bounded degree. Proposition 3.3 shows that if f and g are mainly located on two finite sets R_1 and R_2, then the covariance of f and g decays exponentially in the distance between R_1 and R_2.

From now on, we assume that the set of sites S is endowed with an undirected graph structure, and we denote by d the natural distance on the graph. We will assume not only that the graph is locally finite, but also that the degree of each vertex is uniformly bounded.

$$\forall x \in S\ , \quad |\{y \in S\ ,\ d(x,y) = 1\}| \leq r\ ,$$

where $|\cdot|$ denotes the cardinality of a finite set. Thus the size of the sphere or ball with center x and radius n is uniformly bounded in x, and increases at most geometrically in n.

$$|\{y \in S\ ,\ d(x,y) = n\}| \leq \frac{r}{r-1}(r-1)^n \quad \text{and} \quad |\{y \in S\ ,\ d(x,y) \leq n\}| \leq \frac{r}{r-2}(r-1)^n.$$

Let R be a finite subset of S. We shall use the following upper bounds for the number of vertices at distance n, or at most n from R.

$$|\{x \in S\ ,\ d(x,R) = n\}| \leq |\{y \in S\ ,\ d(x,R) \leq n\}| \leq 2|R|e^{n\rho}\ , \tag{4}$$

with $\rho = \log(r-1)$.

In the case of an amenable graph (e.g. a lattice on $\mathbb{Z}^d$), the ball sizes have a subexponential growth. Therefore, for all $\varepsilon > 0$, there exists c such that :

$$|\{x \in S\ ,\ d(x,R) = n\}| \leq |\{y \in S\ ,\ d(x,R) \leq n\}| \leq ce^{n\varepsilon}.$$

What follows is written in the general case, using (4). It applies to the amenable case replacing ρ by ε, for any $\varepsilon > 0$.

We are going to deal with smooth functions, depending weakly on coordinates away from a fixed finite set R. Indeed, it is not sufficient to consider functions depending only on coordinates in R, because if f is such a function, then for any $t > 0$, $S_t f$ may depend on all coordinates.

Definition 3.1 *Let f be a function from S into $\mathbb{R}$, and R be a finite subset of S. The function f is said to be* mainly located on R *if there exists two constants α and $\beta > \rho$ such that $\alpha > 0$, $\beta > \rho$ and for all $x \in \mathbb{R}$:*

$$\Delta_f(x) \leq \alpha e^{-\beta d(x,R)}. \tag{5}$$

Since $\beta > \rho$, the sum $\sum_x \Delta_f(x)$ is finite. Therefore a function mainly located on a finite set is necessarily smooth.

The system we are considering will be supposed to have finite range interactions in the following sense (cf. Definition 4.17, p. 39 of [**?**]).

Definition 3.2 *A particle system defined by the rates $c_T(\eta, \cdot)$ is said to have finite range interactions if there exists $k > 0$ such that if $d(x, y) > k$:*

1. *$c_T = 0$ for all T containing both x and y ,*
2. *$\gamma(x, y) = 0$.*

The first condition imposes that two coordinates cannot simultaneously change if their distance is larger than k. The second one says that the influence of a site on the transition rates of another site cannot be felt beyond distance k.

Under these conditions, we prove the following covariance inequality.

Proposition 3.3 *Assume (2) and (3). Assume moreover that the process is of finite range. Let R_1 and R_2 be two finite subsets of S. Let β be a constant such that $\beta > \rho$. Let f and g be two functions mainly located on R_1 and R_2, in the sense that there exist positive constants κ_f, κ_g such that,*

$$\Delta_f(x) \leq \kappa_f e^{-\beta d(x,R_1)} \quad \textit{and} \quad \Delta_g(x) \leq \kappa_g e^{-\beta d(x,R_2)}.$$

Then for all positive reals s, t,

$$\sup_{\eta \in X} \left| \mathrm{Cov}_\eta(f(\eta_s), g(\eta_t)) \right| \leq C \kappa_f \kappa_g (|R_1| \wedge |R_2|) e^{D(t+s)} e^{-(\beta-\rho)d(R_1,R_2)} , \tag{6}$$

where

$$D = 2Me^{(\beta+\rho)k} \quad \textit{and} \quad C = \frac{2Be^{\beta k}}{D} \left(1 + \frac{e^{\rho k}}{1 - e^{-\beta+\rho}} \right).$$

Remark. Shashkin [?] obtains a similar inequality for random fields indexed by $\mathbb{Z}^d$.

We now consider a *transitive graph*, such that the group of automorphism acts transitively on S (see chapter 3 of [?]). Namely we need that

- for any x and y in S there exists a in $Aut(S)$, such that $a(x) = y$.
- for any x and y in S and any radius n, there exists a in $Aut(S)$, such that $a(B(x,n)) = B(y,n)$.

Any element a of the automorphism group acts on configurations, functions and measures on $\mathcal{X}$ as follows:

- *configurations:* $a \cdot \eta(x) = \eta(a^{-1}(x))$,
- *functions:* $a \cdot f(\eta) = f(a \cdot \eta)$,
- *measures:* $\int f \, d(a \cdot \mu) = \int (a \cdot f) \, d\mu$.

A probability measure μ on $\mathcal{X}$ is invariant through the group action if $a \cdot \mu = \mu$ for any automorphism a, and we want this to hold for the probability distribution of η_t at all times t. It will be the case if the transition rates are also invariant through the group action. In order to avoid confusions with invariance in the sense of the semigroup (Definition 1.7, p. 10 of [?]), invariance through the action of the automorphism group of the graph will be systematically referred to as "group invariance" in the sequel.

Definition 3.4 *Let G be the automorphism group of the graph. The transition rates $c_T(\eta, \cdot)$ are said to be* group invariant *if for any $a \in G$,*

$$c_{a(T)}(a \cdot \eta, a \cdot \zeta) = c_T(\eta, \zeta).$$

This definition extends in an obvious way that of translation invariance on $\mathbb{Z}^d$-lattices ([?], p. 36).

Remark. Observe that for rates which are both finite range and group invariant, the hypotheses (2) and (3) are trivially satisfied. In that case, it is easy to check that the semi-group $\{S_t \,,\ t \geq 0\}$ commutes with the automorphism group. Thus if μ is a group invariant measure, then so is μS_t for any t (see [?], p. 38). In other terms, if the distribution of η_0 is group invariant, then that of η_t will remain group invariant at all times.

4 Functional CLT

Our functional central limit theorem requires that all coordinates of the interacting particle system $\{\eta_t \,,\ t \geq 0\}$ are identically distributed.

Let $(B_n)_{n\geq 1}$ be an increasing sequence of finite subsets of S such that

$$S = \bigcup_{n=1}^{\infty} B_n, \qquad \lim_{n\to+\infty} \frac{|\partial B_n|}{|B_n|} = 0\;, \tag{7}$$

recall that $|\cdot|$ denotes the cardinality and $\partial B_n = \{x \in B_n\;,\, \exists\, y \notin B_n,\, d(x,y) = 1\}$.

Theorem 4.1 *Let $\mu = \delta_\eta$ be a Dirac measure where $\eta \in \mathcal{X}$ fulfills $\eta(x) = \eta(y)$ for any $x, y \in S$. Suppose that the transition rates are group invariant. Suppose moreover that the process is of finite range, monotone and fulfilling the requirements of Corollary 2.4. Let $(B_n)_{n\geq 1}$ be an increasing sequence of finite subsets of S fulfilling (7). Then the sequence of processes*

$$\left\{ \frac{N_t^{B_n} - \mathbb{E}_\mu N_t^{B_n}}{\sqrt{|B_n|}}\;,\; t \geq 0 \right\}, \qquad \textit{for } n = 1, 2, \ldots$$

converges in $D([0,T])$ as n tends to infinity, to a centered Gaussian, vector valued process $(B(t,w))_{t\geq 0,\, w\in W}$ with covariance function Γ defined, for w, $w' \in W$, by

$$\Gamma_\mu(s,t)(w,w') = \sum_{x\in S} \mathrm{Cov}_\mu\left(\mathbb{I}_w(\eta_s(x)), \mathbb{I}_{w'}(\eta_t(x))\right).$$

Remark. One may wonder wether such results can extend under more general initial distributions. The point is that the covariance inequality do not extend simply by integration with respect to deterministic configurations. We are thankful to Pr. Penrose for stressing our attention on this important restriction. Monotonicity allows to get ride of this restriction.

5 Examples of graphs

Besides the classical lattice graphs in $\mathbb{Z}^d$ and their groups of translations, which are considered by most authors (see [**?**, **?**, **?**]), our setting applies to a broad range of graphs. We propose some simple examples of automorphisms on trees, which give rise to a large variety of non classical situations.

The simplest example corresponds to regular trees defined as follows. Consider the non-commutative free group S with finite generator set G. Impose that each generator g is its own inverse ($g^2 = 1$). Now consider S as a graph, such that x and y are connected if and only if there exists $g \in G$ such that $x = yg$. Note that S is a regular tree of degree equal to the cardinality r of G. The size of spheres is exponential: $|\{y\;,\, d(x,y) = n\}| = r^n$. Now consider the group action of S on itself: $x \cdot y = xy$: this action is transitive on S (take $a = yx$).

From this basic example it is possible to get a large class of graphs by adding relations between generators; for example take the tree of degree 4, denote by a, b, c, and d the

generators, and add the relation $ab = c$. Then, the corresponding graph is a regular tree of degree 4 were nodes are replaced by tetrahedrons. The spheres do not grow at rate 4^n: $|\{y\,,\, d(x,y) = n\}| = 4 \cdot 3^{n/2}$ if n is even and $|\{y\,,\, d(x,y) = n\}| = 6 \cdot 3^{(n-1)/2}$ if n is odd.

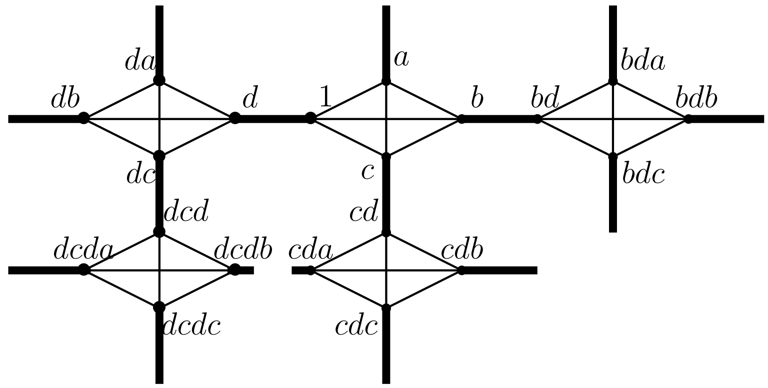

Figure 1: Graph structure of the tree with tetrahedron cells. The graph consists in a regular tree of degree 4 (bold lines), where nodes have been replaced by tetrahedrons. Automorphisms in this graph correspond to composition of automorphisms exchanging couples of branches of the tree (action of generator a for example) and displacements in the subjacent regular tree.

6 CLT for hitting times

In this section we consider the case where W is ordered, the process is monotone and satisfies the assumptions in Theorem 4.1, the initial condition is fixed and f is an increasing function from W to $\mathbb{R}$. In the reliability interpretation, $f(w)$ measures a level of degradation for a component in state w. The total degradation of the system in state η will be measured by the sum $\sum_{x \in B_n} f(\eta(x))$. So we shall focus on the process $D^{(n)} = \{D_t^{(n)}\,,\, t \geq 0\}$, where $D_t^{(n)} = D_t^{B_n}$ is the total degradation of the system at time t on the set $R = B_n$:

$$D_t^{(n)} = \sum_{x \in B_n} f(\eta_t(x)).$$

It is natural to consider the instants at which $D_t^{(n)}$ reaches a prescribed level of degradation. Let $k = (k(n))$ be a sequence of real numbers. Our main object is the *failure time* T_n, defined as:

$$T_n = \inf\{t \geq 0\,,\; D_t^{(n)} \geq k(n)\}.$$

In the particular case where $W = \{\text{working, failed}\}$ (binary components), and f is the indicator of a failed component, then $D_t^{(n)}$ simply counts the number of failed components at time t, and our system is a so-called "k-out-of-n" system [**?**].

Let w_0 be a particular state (in the reliability w_0 could be the "perfect state" of an undergrade component). Let η be the constant configuration where all components are in the perfect state w_0, for all $x \in S$. Our process starts from that configuration η, which is obviously group invariant. We shall denote by $m(t)$ (respectively, $v(t)$) the expectation (resp., the variance) of the degradation at time t for one component.

$$m(t) = \mathbb{E}[f(\eta_t(x)) \,|\, \eta_0 = \eta] \;, \qquad v(t) = \lim_{n\to\infty} \frac{\operatorname{Var} D_t^{(n)}}{|B_n|}.$$

These expressions do not depend on $x \in S$, due to group invariance.

The average degradation $D_t^{(n)}/|B_n|$ converges in probability to its expectation $m(t)$. We shall assume that $m(t)$ is strictly increasing on the interval $[0, \tau]$, with $0 < \tau \leq +\infty$ (the degradation starting from the perfect state increases on average). Mathematically, one can assume that the states are ranked in increasing order, the perfect state being the lowest. This yields a partial order on configurations. If the rates are such that the interacting particle system is monotone (see [**?**]), then the average degradation increases. In the reliability interpretation, assuming monotonicity is quite natural: it amounts to saying that the rate at which a given component jumps to a more degraded state is higher if its surroundings are more degraded.

We consider a "mean degradation level" α, such that $m(0) < \alpha < m(\tau)$. Assume the threshold $k(n)$ is such that:

$$k(n) = \alpha |B_n| + o(\sqrt{|B_n|}).$$

Theorem 4.1 shows that the degradation process $D^{(n)}$ should remain at distance $O(\sqrt{|B_n|})$ from the deterministic function $|B_n|\, m$. Therefore it is natural to expect that T_n is at distance $O(1/\sqrt{|B_n|})$ from the instant t_α at which $m(t)$ crosses α:

$$t_\alpha = \inf\{t,\ m(t) = \alpha\}.$$

Theorem 6.1 *Under the above hypotheses,*

$$\sqrt{|B_n|}\,(T_n - t_\alpha) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0, \sigma_\alpha^2),$$

with:

$$\sigma_\alpha^2 = \frac{v(t_\alpha)}{(m'(t_\alpha))^2}.$$

7 CLT for weakly dependent random fields

As in section 4, we consider a transitive graph $\mathcal{G} = (S, E)$, where S is the set of vertices and $E \subset \Big\{ \{x, y\},\ x, y \in\ S,\ x \neq y \Big\}$ the set of edges. For a transitive graph, the degree r of each vertex is constant (cf. Lemma 1.3.1 in Godsil and Royle [?]).

For any x in S and for any positive integer n, we denote by $B(x, n)$ the open ball of S centered at x, with radius n:

$$B(x, n) = \{y \in S,\ \ d(x, y) < n\}.$$

The cardinality of the ball $B(x, n)$ is constant in x and bounded as follows.

$$\sup_{x \in S} |B(x, n)| \leq 2r^n = 2e^{n\rho} =: \kappa_n, \tag{8}$$

where $\rho = \ln(\max(r, 4) - 1)$: compare with formula (4).

Let $Y = (Y_x)_{x \in S}$ be a real valued random field. We will measure covariances between coordinates of Y on two distant sets R_1 and R_2 through Lipschitz functions (see [?]). A Lipschitz function is a real valued functions f defined on $\mathbb{R}^n$ for some positive integer n, for which

$$\operatorname{Lip} f := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\sum_{i=1}^n |x_i - y_i|} < \infty.$$

We will assume the the random field Y satisfies the following covariance inequality: for any positive real δ, for any disjoint finite subsets R_1 and R_2 of S and for any Lipschitz functions f and g defined respectively on $\mathbb{R}^{|R_1|}$ and $\mathbb{R}^{|R_2|}$, there exists a positive constant C_δ (not depending on f g, R_1 and R_2) such that

$$|\operatorname{Cov}\left(f(Y_x,\ x \in R_1), g(Y_x,\ x \in R_2)\right| \leq C_\delta \operatorname{Lip} f \operatorname{Lip} g\ \left(|R_1| \wedge |R_2|\right) \exp\left(-\delta d(R_1, R_2)\right). \tag{9}$$

For any finite subset R of S, let $Z(R) = \sum_{x \in R} Y_x$. Let $(B_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of S such that $|B_n|$ goes to infinity with n. Our purpose in this section is to establish a central limit theorem for $Z(B_n)$, suitably normalized. We suppose that $(Y_x)_{x \in S}$ is a weakly dependent random field according to the covariance inequality (9).

In Proposition 7.1 below we prove that, as in the independent setting, a central limit theorem holds as soon as $\operatorname{Var} Z(B_n)$ behaves, as n goes to infinity, like $|B_n|$ (cf. Condition (11) below). So the purpose of Proposition 7.2 is to study the behavior of $\operatorname{Var} Z(B_n)$. We prove that the limit (11) holds under two additional conditions. The first one supposes that the cardinality of ∂B_n is asymptotically negligible compared to $|B_n|$ (cf. Condition (7) in section 4); the second condition supposes an invariance by the automorphisms of the group $\mathcal{G}$, of the joint distribution (Y_x, Y_y) for any two vertices x and y. More precisely we need to have Condition (10) below,

$$\operatorname{Cov}(Y_x, Y_y) = \operatorname{Cov}(Y_{a(x)}, Y_{a(y)}), \tag{10}$$

for any automorphism a of $\mathcal{G}$.

In order to prove Proposition 7.1, we shall use some estimations of Bolthausen [?] that yield a central limit theorem for stationary random fields on $\mathbb{Z}^d$ under mixing conditions. Recall that the mixing coefficients used there are defined as follows, noting by $\mathcal{A}_R$ the σ-algebra generated by $(Y_x,\, x \in R)$,

$$\alpha_{k,l}(n) = \sup\{|\mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2)|,\, A_i \in \mathcal{A}_{R_i},\, |R_1| \leq k,\, |R_2| \leq l,\, d(R_1, R_2) \geq n\},$$

for $n \in \mathbb{N}$ and $k, l \in \mathbb{N} \cup \infty$,

$$\rho(n) = \sup\{|\mathrm{Cov}(Z_1, Z_2)|,\, Z_i \in L_2(\mathcal{A}_{\{\rho_i\}}),\, \|Z_i\|_2 \leq 1,\, d(\rho_1, \rho_2) \geq n\}.$$

Under suitable decay of $(\alpha_{k,l}(n))_n$ or of $(\rho(n))_n$, Bolthausen [?] proved a central limit theorem for stationary random fields on $\mathbb{Z}^d$, using an idea of Stein. In our case, instead of using those mixing coefficients, we describe the dependence structure of the random fields $(Y_x)_{x\in S}$ in terms of the gap between two Lipschitz transformations of two disjoint blocks (the covariance inequality (9) above). Those manners of describing the dependence of random fields are quite different. As one may expect, the techniques of proof will be different as well (see section 8).

Proposition 7.1 *Let $\mathcal{G} = (S, E)$ be a transitive graph. Let $(B_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of S such that $|B_n|$ goes to infinity with n. Let $(Y_x)_{x\in S}$ be a real valued random field, satisfying (9). Suppose that, for any $x \in S$, $\mathbb{E}Y_x = 0$ and $\sup_{x\in S} \|Y_x\|_\infty < \infty$. If, there exists a finite real number σ^2 such that*

$$\lim_{n\to\infty} \frac{\mathrm{Var}\, Z(B_n)}{|B_n|} = \sigma^2, \tag{11}$$

then the quantity $\dfrac{Z(B_n)}{\sqrt{|B_n|}}$ converges in distribution to a centered normal law with variance σ^2.

Proposition 7.2 *Let $\mathcal{G} = (S, E)$ be a transitive graph. Let $(Y_x)_{x\in S}$ be a centered real valued random field, with finite variance. Suppose that the conditions (9) and (10) are satisfied. Let $(B_n)_n$ be a sequence of finite and increasing sets of S fulfilling (7). Then*

$$\sum_{z\in S} |\mathrm{Cov}(Y_0, Y_z)| < \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{1}{|B_n|} \mathrm{Var}\, Z(B_n) = \sum_{z\in S} \mathrm{Cov}(Y_0, Y_z).$$

8 Proofs

8.1 Proof of Proposition 3.3

Let Γ denote the matrix $(\gamma(x,y))_{x,y\in S}$, and let it operate on the right on the space of summable series $\ell_1(S)$ indexed by the denumerable set S:

$$u = (u(x))_{x\in S} \mapsto \Gamma u = (\Gamma u(y))_{y\in S}\,,$$

with :

$$\Gamma u(y) = \sum_{x \in S} u(x)\, \gamma(x, y).$$

(We have followed Liggett's [?] choice of denoting by Γu the product of u by Γ on the right.) Thanks to hypothesis (3), this defines a bounded operator of $\ell_1(S)$, with norm M. Thus for all $t \geq 0$, the exponential of $t\Gamma$, is well defined, and gives another bounded operator of $\ell_1(S)$:

$$\exp(t\Gamma)u = \sum_{n=0}^{\infty} \frac{t^n \Gamma^n u}{n!}.$$

If f is a smooth function, then $\Delta_f = (\Delta_f(x))_{x \in S}$, is an element of $\ell_1(S)$. Applying $\exp(t\Gamma)$ to Δf provides a control on $S_t f$ as shows the following proposition (cf. Theorem 3.9 of [?]).

Proposition 8.1 *Assume* (2) *and* (3)*. Let f be a smooth function. Then,*

$$\Delta_{S_t f} \leq \exp(t\Gamma)\Delta_f. \tag{12}$$

It follows immediately that if f is a smooth function then $S_t f$ is also smooth and:

$$|||S_t f||| \leq e^{tM}\, |||f|||\,,$$

because the norm of $\exp(t\Gamma)$ operating on $\ell_1(S)$ is e^{tM}.
A similar bound for covariances will be our starting point (cf. Proposition 4.4, p. 34 of [?]).

Proposition 8.2 *Assume* (2) *and* (3)*. Then for any smooth functions f and g and for all $t \geq 0$, one has,*

$$\|S_t fg - (S_t f)(S_t g)\| \leq \sum_{y,z \in S} \left(\sum_{T \ni y,z} c_T \right) \int_0^t (\exp(\tau\Gamma)\Delta_f)(y)(\exp(\tau\Gamma)\Delta_g)(z)\, d\tau. \tag{13}$$

In terms of the process $\{\eta_t\,,\, t \geq 0\}$, the left member of (13) is the uniform bound for the covariance between $f(\eta_t)$ and $g(\eta_t)$.

$$\|S_t fg - (S_t f)(S_t g)\| = \sup_{\eta \in X} \Big| \mathbb{E}_\eta[f(\eta_t)g(\eta_t)] - \mathbb{E}_\eta[f(\eta_t)]\mathbb{E}_\eta[g(\eta_t)] \Big|.$$

A slight modification of (13) gives a bound on the covariance of $f(\eta_s)$ with $g(\eta_t)$, for $0 \leq s \leq t$. From now on, we shall denote by Cov_η covariances relative to the distribution of $\{\eta_t\,,\ t \geq 0\}$, starting at $\eta_0 = \eta$:

$$\mathrm{Cov}_\eta(f(\eta_s), g(\eta_t)) = \mathbb{E}_\eta[f(\eta_s)g(\eta_t)] - \mathbb{E}_\eta[f(\eta_s)]\mathbb{E}_\eta[g(\eta_t)].$$

Corollary 8.3 *Assume* (2) *and* (3). *Let f and g be two smooth functions. Then for all s and t such that $0 \leq s \leq t$,*

$$\sup_{\eta \in X} \left| \mathrm{Cov}_\eta(f(\eta_s), g(\eta_t)) \right| \leq \sum_{y,z \in S} \left(\sum_{T \ni y,z} c_T \right) \int_0^s (\exp(\tau\Gamma)\Delta_f)(y)(\exp(\tau\Gamma)\Delta_{S_{t-s}g})(z)\, d\tau. \tag{14}$$

Proof of Corollary 8.3. We have, using the semigroup property,

$$\mathbb{E}_\eta[f(\eta_s)g(\eta_t)] = \mathbb{E}_\eta[f(\eta_s)\mathbb{E}[g(\eta_t) \mid \eta_s]] = \mathbb{E}_\eta[f(\eta_s)S_{t-s}g(\eta_s)] = S_s(fS_{t-s}g)(\eta).$$

Also,

$$\mathbb{E}_\eta[g(\eta_t)] = S_t g(\eta) = S_s(S_{t-s}g)(\eta).$$

Applying (13) at time s to f and $S_{t-s}g$, yields the result. $\square$

In order to apply (14) to functions mainly located on finite sets, we shall need to control the effect of $\exp(t\Gamma)$ on a sequence $(\Delta_f(x))$ satisfying (5). This will be done through the following technical lemma.

Lemma 8.4 *Suppose that the process is of finite range. Let R be a finite set of sites. Let $u = (u(x))_{x \in S}$ be an element of $\ell_1(S)$. If for all $x \in S$, $u(x) \leq \alpha e^{-\beta d(x,R)}$, with $\alpha > 0$ and $\beta > \rho$, then for all $y \in S$,*

$$|(\exp(t\Gamma)u)(y)| \leq \alpha \exp(2tMe^{(\beta+\rho)k})\, e^{-\beta d(y,R)}.$$

This lemma, together with Proposition 8.1, justifies Definition 3.1. Indeed, if f is mainly located on R, then by (12) and Lemma 8.4, $S_t f$ is also mainly located on R, and the rate of exponential decay β is the same for both functions.

Proof of Lemma 8.4. Recall that

$$\Gamma u(y) = \sum_{x \in S} u(x)\gamma(x,y).$$

Observe that if $\gamma(x,y) > 0$, then the distance from x to y must be at most k and thus the distance from x to R is at least $d(y,R) - k$. If $u(x) \leq \alpha e^{-\beta d(x,R)}$ then:

$$\Gamma u(y) \leq 2\alpha e^{\rho k} e^{-\beta(d(y,R)-k)} M = 2\alpha e^{(\beta+\rho)k} M e^{-\beta d(y,R)}.$$

Hence by induction,

$$\Gamma^n u(y) \leq \alpha 2^n e^{(\beta+\rho)kn} M^n e^{-\beta d(y,R)}.$$

The result follows immediately. $\square$

Together with (14), Lemma 8.4 will be the key ingredient in the proof of our covariance inequality.

End of the proof of Proposition 3.3. Being mainly located on finite sets, the functions f and g are smooth. By (14), the covariance of $f(\eta_s)$ and $g(\eta_t)$ is bounded by $M(s,t)$ with:

$$M(s,t) = \sum_{y,z \in S} \left(\sum_{T \ni y,z} c_T \right) \int_0^s (\exp(\tau\Gamma)\Delta_f)(y)(\exp(\tau\Gamma)\Delta_{S_{t-s}g})(z)\, d\tau.$$

Let us apply Lemma 8.4 to Δ_f and $\Delta_{S_{t-s}g}$.

$$(\exp(\tau\Gamma)\Delta_f)(y) \le \kappa_f \exp(\tau M e^{(\beta+\rho)k}) e^{-\beta d(y,R_1)} = \kappa_f e^{D\tau} e^{-\beta d(y,R_1)}. \tag{15}$$

The last bound, together with (12), gives

$$\Delta_{S_{t-s}g}(x) \le (\exp((t-s)\Gamma)\Delta_g)(x) \le \kappa_g e^{D(t-s)} e^{-\beta d(x,R_2)}.$$

Therefore :

$$(\exp(\tau\Gamma)\Delta_{S_{t-s}g})(z) \le \kappa_g e^{D(\tau+t-s)} e^{-\beta d(z,R_2)}. \tag{16}$$

Inserting the new bounds (15) and (16) into $M(s,t)$, we obtain

$$M(s,t) \le \sum_{y,z \in S} \left(\sum_{T \ni y,z} c_T \right) \kappa_f \kappa_g e^{-\beta(d(y,R_1)+d(z,R_2))} \int_0^s e^{D(2\tau+t-s)}\, d\tau.$$

Now if $d(y,z) > k$ and $y,z \in T$, then c_T is null by Definition 3.2. Remember moreover that by hypothesis (2):

$$B = \sup_{u \in S} \sum_{T \ni u} c_T < \infty.$$

Therefore :

$$M(s,t) \le \kappa_f \kappa_g \frac{B e^{D(s+t)}}{2D} \sum_{y \in S} \sum_{d(y,z) \le k} e^{-\beta(d(y,R_1)+d(z,R_2))}. \tag{17}$$

In order to evaluate the last quantity, we have to distinguish two cases.

- If $d(R_1,R_2) \le k$, then

$$\begin{aligned}
\sum_{y \in S} \sum_{d(y,z) \le k} e^{-\beta(d(y,R_1)+d(z,R_2))} &\le 2e^{\rho k} \sum_{y \in S} e^{-\beta d(y,R_1)} \\
&\le 2e^{\rho k} \sum_{n \in \mathbb{N}} \sum_{y \in S} e^{-\beta d(y,R_1)} \mathbb{I}_{d(y,R_1)=n} \\
&\le 4|R_1| e^{\rho k} \sum_{n=0}^{\infty} e^{(\rho-\beta)n} \\
&\le \frac{4|R_1| e^{\rho k}}{1-e^{-(\beta-\rho)}} \\
&\le |R_1| \frac{4e^{(\rho+\beta)k}}{1-e^{-(\beta-\rho)}} e^{-\beta d(R_1,R_2)} \\
&\le |R_1| \frac{4e^{(\rho+\beta)k}}{1-e^{-(\beta-\rho)}} e^{-(\beta-\rho) d(R_1,R_2)}
\end{aligned}$$

• If $d(R_1,R_2) > k$, then we have, noting that $d(y,R_1)+d(z,R_2) \geq d(R_1,R_2) - d(y,z)$ and that $d(y,z) \leq k$,

$$\begin{aligned}
&\sum_{y\in S}\sum_{d(y,z)\leq k} e^{-\beta(d(y,R_1)+d(z,R_2))} \\
&\quad\leq \sum_{d(y,R_1)\leq d(R_1,R_2)-k}\ \sum_{d(y,z)\leq k} e^{-\beta(d(R_1,R_2)-k)} + \sum_{d(y,R_1)\geq d(R_1,R_2)-k}\ \sum_{d(y,z)\leq k} e^{-\beta d(y,R_1)} \\
&\quad\leq 4|R_1|\, e^{\rho(d(R_1,R_2)-k)} e^{\rho k} e^{-\beta(d(R_1,R_2)-k)} + 4|R_1| e^{\rho k} \sum_{n\geq d(R_1,R_2)-k} e^{(\rho-\beta)n} \\
&\quad\leq 4|R_1|\, e^{\beta k}\left(1+\frac{1}{1-e^{-(\beta-\rho)}}\right) e^{-(\beta-\rho)d(R_1,R_2)}.
\end{aligned}$$

By inserting the latter bound into (17), one obtains,

$$M(s,t) \leq C\kappa_f\kappa_g |R_1| e^{D(t+s)} e^{-(\beta-\rho)d(R_1,R_2)}\ ,$$

with :

$$C = \frac{2B}{D} e^{\beta k}\left(1+\frac{e^{\rho k}}{1-e^{-\beta+\rho}}\right). \qquad \square$$

The covariance inequality (6) implies that the covariance between two functions essentially located on two distant sets decays exponentially with the distance of those two sets, whatever the instants at which it is evaluated. However the upper bound increases exponentially fast with s and t. In the case where the process $\{\eta_t\,,\ t\geq 0\}$ converges at exponential speed to its equilibrium, it is possible to give a bound that increases only in $t-s$, thus being uniform in t for the covariance at a given instant t.

8.2 Proof of Theorem 4.1

8.2.1 Finite dimensional laws

Let $\mathcal{G} = (S,E)$ be a transitive graph and $Aut(\mathcal{G})$ be the automorphism group of $\mathcal{G}$. Let μ be a probability measure on $\mathcal{X}$ invariant through the automorphism group action. Let $(\eta_t)_{t\geq 0}$ be an interacting particle system fulfilling the requirements of Theorem 4.1. Recall that $\{S_t\,,\ t\geq 0\}$ denotes the semigroup and μS_t the distribution of η_t, if the distribution of η_0 is μ.

Proposition 8.5 *Let $(B_n)_n$ be an increasing sequence of finite subsets of S fulfilling* (7). *Let assumptions of Theorem 4.1 hold. Then for any fixed positive real numbers $t_1 \leq t_2 \leq \cdots \leq t_k$, the random vector*

$$\frac{1}{\sqrt{|B_n|}}\left(N^{B_n}_{t_1} - \mathbb{E}_\mu N^{B_n}_{t_1}, N^{B_n}_{t_2} - \mathbb{E}_\mu N^{B_n}_{t_2}, \ldots, N^{B_n}_{t_k} - \mathbb{E}_\mu N^{B_n}_{t_k}\right)$$

converges in distribution, as n tends to infinity, to a centered Gaussian vector with covariance matrix $(\Gamma_\mu(t_i,t_j))_{1\le i,j\le k}$.

Proof of Proposition 8.5. We will only study the convergence in distribution of the vector

$$\frac{1}{\sqrt{|B_n|}}\left(N_{t_1}^{B_n}-\mathbb{E}_\mu N_{t_1}^{B_n}, N_{t_2}^{B_n}-\mathbb{E}_\mu N_{t_2}^{B_n}\right) ,$$

the general case being similar. For $i=1,2$, we denote by $\alpha_i=(\alpha_i(w))_{w\in W}$ two fixed vectors of $\mathbb{R}^{|W|}$. We have, denoting by $\cdot$ the usual scalar product,

$$\begin{aligned}
&\frac{1}{\sqrt{|B_n|}}\sum_{i=1}^{2}\alpha_i\cdot\left(N_{t_i}^{B_n}-\mathbb{E}_\mu N_{t_i}^{B_n}\right)\\
&\quad=\frac{1}{\sqrt{|B_n|}}\sum_{x\in B_n}\left(\sum_{i=1}^{2}\left(\sum_{w\in W}\alpha_i(w)(\mathbb{I}_w(\eta_{t_i}(x))-\mathbb{P}_\mu(\eta_{t_i}(x)=w))\right)\right)\\
&\quad=\frac{1}{\sqrt{|B_n|}}\sum_{x\in B_n}Y_x,
\end{aligned}$$

where $(Y_x)_{x\in S}$ is the random field defined by

$$Y_x=\sum_{i=1}^{2}\left(\sum_{w\in W}\alpha_i(w)(\mathbb{I}_w(\eta_{t_i}(x))-\mathbb{P}_\mu(\eta_{t_i}(x)=w))\right)=:F_1(\eta_{t_1}(x))+F_2(\eta_{t_2}(x)). \qquad (18)$$

The purpose is then to prove a central limit theorem for the sum $\sum_{x\in B_n}Y_x$. For this, we shall study the nature of the dependence of $(Y_x)_{x\in S}$.

Let R_1 and R_2 be two finite and disjoints subsets of S. Let k_1 and k_2 be two real valued functions defined respectively on $\mathbb{R}^{|R_1|}$ and $\mathbb{R}^{|R_2|}$. Let K_1, K_2 be two real valued functions, defined respectively on W^{R_1} and W^{R_2}, by

$$K_j(\nu,\eta)=k_j(F_1(\nu(x))+F_2(\eta(x)),\ x\in R_j),\quad j=1,2.$$

Let $\mathcal{L}$ be the class of real valued Lipschitz functions f defined on $\mathbb{R}^n$, for some positive integer n, for which

$$\mathrm{Lip}\, f:=\sup_{x\neq y}\frac{|f(x)-f(y)|}{\sum_{i=1}^n|x_i-y_i|}<\infty.$$

We assume that k_1 and k_2 belong to $\mathcal{L}$. Recall that

$$\mathrm{Cov}_\eta(k_1(Y_x,\,x\in R_1),k_2(Y_x,\,x\in R_2)) \;=\; \mathrm{Cov}_\eta\left(K_1(\eta_{t_1},\eta_{t_2}),K_2(\eta_{t_1},\eta_{t_2})\right)$$

But

$$|K_1(\eta_{t_1},\eta_{t_2})-K_1(\eta'_{t_1},\eta_{t_2})|\le 4\mathrm{Lip}\,k_1\sum_{w\in W}|\alpha_1(w)|\sum_{x\in R_1}|\eta_{t_1}(x)-\eta'_{t_1}(x)|$$

Denote $A_1(W) = 4\mathrm{Lip}\, k_1 \sum_{w\in W} |\alpha_1(w)|$. Then, the functions

$$\eta_{t_1} \longrightarrow (\mathrm{Lip}\, k_1) A_1(W) \sum_{x\in R_1} \eta_{t_1}(x) \pm\ K_1(\eta_{t_1}, \eta_{t_2})$$

are increasing. Hence, the functions

$$G_1^{\pm} : (\eta_{t_1}, \eta_{t_2}) \longrightarrow \mathrm{Lip}\, k_1 \sum_{x\in R_1} (A_1(W)\eta_{t_1}(x) + A_2(W)\eta_{t_2}(x)) \pm\ K_1(\eta_{t_1}, \eta_{t_2})$$

are increasing coordinate by coordinate. This also holds for,

$$G_2^{\pm} : (\eta_{t_1}, \eta_{t_2}) \longrightarrow \mathrm{Lip}\, k_2 \sum_{x\in R_2} (A_1(W)\eta_{t_1}(x) + A_2(W)\eta_{t_2}(x)) \pm\ K_2(\eta_{t_1}, \eta_{t_2}).$$

Under assumptions of Theorem 2.3 and of its Corollary 2.4, the vector (η_{t_1}, η_{t_2}) has positive correlation so that

$$\mathrm{Cov}_\eta(G_1^{\pm}(\eta_{t_1}, \eta_{t_2}), G_2^{\pm}(\eta_{t_1}, \eta_{t_2})) \geq 0.$$

This gives

$$\begin{aligned}
&|\mathrm{Cov}_\eta(k_1(Y_x,\, x\in R_1), k_2(Y_x,\, x\in R_2))| \\
&\leq \mathrm{Lip}\, k_1 \mathrm{Lip}\, k_2 \sum_{x\in R_1}\sum_{y\in R_2} \mathrm{Cov}_\eta(A_1(W)\eta_{t_1}(x) + A_2(W)\eta_{t_2}(x), A_1(W)\eta_{t_1}(y) + A_2(W)\eta_{t_2}(y)).
\end{aligned}$$

From this bilinear formula, we now apply Proposition 3.3 and obtain the following covariance inequality: for finite subsets R_1 and R_2 of S, we have letting $\delta = \beta - \rho$,

$$|\mathrm{Cov}_\eta\left(K_1(\eta_{t_1}, \eta_{t_2}), K_2(\eta_{t_1}, \eta_{t_2})\right)| \leq C_\delta \mathrm{Lip}\, k_1 \mathrm{Lip}\, k_2 \left(|R_1| \wedge |R_2|\right) \exp\left(-\delta d(R_1, R_2)\right),$$

where C_δ is a positive constant depending on β and not depending on R_1, R_2, k_1 and k_2. We then deduce from Proposition 7.1 that $\frac{1}{\sqrt{|B_n|}} \sum_{x\in B_n} Y_x$ converges in distribution to a centered normal law as soon as the quantity $\mathrm{Var}_\mu(\sum_{x\in B_n} Y_x)/|B_n|$ converges as n tends to infinity to a finite number σ^2. This variance converges if the requirements of Proposition 7.2 are satisfied. For this, we first check the condition of invariance (10):

$$\mathrm{Cov}_\mu(Y_x, Y_y) = \mathrm{Cov}_\mu(Y_{a(x)}, Y_{a(y)}),$$

for any automorphism a of $\mathcal{G}$ and for Y_x as defined by (18). We recall that the initial distribution is a Dirac distribution on the configuration η. Then it has positive correlations. We have supposed that $\eta(x) = \eta(y)$ for all $x, y \in S$, hence $a\cdot\mu = \mu$ and the group invariance property of the transition rates proves that $\mu = \delta_\eta$ fulfills (19) below and then (10) will hold. Condition (19) is true thanks to the following estimations valid for any suitable real

valued functions f and g,

$$\begin{aligned}
&\mathbb{E}_\mu(f(\eta_{t_1})g(\eta_{t_2}))\\
&\quad= \int d\mu(\eta) S_{t_1}\left(f S_{t_2-t_1} g\right)(\eta)\\
&\quad= \int d\mu(\eta)\, a\cdot S_{t_1}\left(f S_{t_2-t_1} g\right)(\eta) \quad \text{since} \quad a\cdot\mu=\mu\\
&\quad= \int d\mu(\eta) S_{t_1}\left((a\cdot f) S_{t_2-t_1}(a\cdot g)\right)(\eta) \quad \text{since} \quad a\cdot(S_s f) = S_s(a\cdot f)\\
&\quad= \mathbb{E}_\mu((a\cdot f)(\eta_{t_1})(a\cdot g)(\eta_{t_2})) = \mathbb{E}_\mu(f(a\cdot\eta_{t_1})g(a\cdot\eta_{t_2})).
\end{aligned} \tag{19}$$

Hence Proposition 7.2 applies and gives

$$\begin{aligned}
\sigma^2 &= \sum_{z\in S} \mathrm{Cov}_\mu(Y_0, Y_z)\\
&= \sum_{i,j=1}^{2} \sum_{w,w'\in W} \alpha_i(w)\alpha_j(w') \sum_{z\in S} \mathrm{Cov}_\mu\left(\mathbb{I}_w(\eta_{t_i}(0)), \mathbb{I}_{w'}(\eta_{t_i}(z))\right)\\
&= \sum_{i,j=1}^{2} \alpha_i^t \Gamma_\mu(t_i,t_j)\alpha_j,
\end{aligned}$$

where $\Gamma_\mu(t_i,t_j)$ is the covariance matrix as defined in Theorem 4.1; with this we complete the proof of Proposition 8.5.

8.2.2 Tightness

First we establish covariance inequalities for the counting process. Denote $g_{s,t,w}(\eta,y) = \mathbb{I}_w(\eta_t(y)) - \mathbb{I}_w(\eta_s(y))$ and for any multi-index $\mathbf{y} = (y_1,\ldots,y_u) \in S^u$, for any state vector $\mathbf{w} = (w_1,\ldots,w_u) \in W^u$, $\Pi_{\mathbf{y},\mathbf{w}} = \prod_{\ell=1}^u g_{s,t,w_\ell}(\eta,y_\ell)$. Following (6), for $\beta > \rho$, for any r-distant finite multi-indices $\mathbf{y}\in S^u$ and $\mathbf{z}\in S^v$, for any times $0\le s\le t\le T$ and for any state vectors $\mathbf{w}\in W^u$ and $\mathbf{w'}\in W^v$

$$|\mathrm{Cov}_\eta\left(\Pi_{\mathbf{y},\mathbf{w}}, \Pi_{\mathbf{z},\mathbf{w'}}\right)| \le 4C(u\wedge v)e^{2DT}e^{-(\beta-\rho)r} \equiv c_0(u\wedge v)e^{-cr}, \tag{20}$$

for $c=\beta-\rho$ and $c_0 = \dfrac{4Be^{2DT}e^{-(\beta-\rho)r}(2-e^{-c})}{Me^{\rho k}(1-e^{-c})}$.

Lemma 8.6 *There exist $\delta_0>0$ and $K_\Omega>0$ such that for $|s-t|<\delta_0$:*

$$|\mathrm{Cov}_\eta\left(\Pi_{\mathbf{x},\mathbf{w}}, \Pi_{\mathbf{y},\mathbf{w'}}\right)| \le K_\Omega |t-s|. \tag{21}$$

Proof. Denote $f(\eta) = \mathbb{I}_w(\eta(x))$ then $g_{t+h,t,w}(\eta,x) = S_h f(\eta_t) - f(\eta_t)$; the properties of the generator Ω imply that

$$\lim_{h\to 0} \frac{S_h f(\eta) - f(\eta)}{h} = \Omega f(\eta)$$

But

$$\begin{aligned}|\Omega f(\eta)| &\leq \sum_{T\subset S}\sum_{\zeta\in W^T} c_T(\eta,\zeta)|f(\eta^\zeta)-f(\eta)| \\ &\leq \sum_{T\subset S, x\in T} c_T(\eta) \leq \sum_{T\subset S, x\in T} c_T \leq C_\Omega\end{aligned}$$

so that for $h>0$ tending to zero

$$|g_{s,s+h,w}(\eta,x)| \leq C_\Omega h + o(h)$$

Because Ω is group invariant, the remainder term is uniform with respect to index x, so that we find convenient δ_0 and K_Ω uniformly with respect to location. $\square$

From inequality (20) and lemma 8.6, we deduce the following moment inequality:

Proposition 8.7 *Choose l and c such that $\rho(2l-1)<c$. For (s,t) such that $|t-s|<\delta_0\wedge c_0e^c/K_\Omega$:*

$$\mathbb{E}(N_t^{B_n}-N_s^{B_n})^{2l} \leq \frac{(4l-2)!(c_0e^{2c})^{\frac{\rho l}{c}}}{(2l)!(2l-1)!} \left(\frac{2^{2l}(2l)!(c_0e^{2c})^{\frac{\rho(l-1)}{c}}}{c_1}|B_n|^{1-l}(K_\Omega|t-s|)^{1-\frac{\rho(2l-1)}{c}} + \left(\frac{8}{c_1}\right)^l (K_\Omega|t-s|)^{l-\frac{\rho l}{c}}\right), \quad (22)$$

where $c_1=\rho\wedge(c-\rho(2l-1))$.

Proof. Recall that $N_t^{B_n}-N_s^{B_n} = \frac{1}{\sqrt{|B_n|}}\sum_{x\in B_n} g_{s,t,w}(\eta,x)$. Note that the value of $\Pi_{\mathbf{x}}$ does not depend on the order of the elements $x_1,\ldots,x_L$. The index $\mathbf{x}$ is said to split into $\mathbf{y}=(y_1,\ldots,y_M)$ and $\mathbf{z}=(z_1,\ldots,z_{L-M})$ if one can write $y_1=x_{\sigma(1)},\ldots,y_M=x_{\sigma(M)}$ and $z_1=x_{\sigma(M+1)},\ldots,z_{L-M}=x_{\sigma(L)}$ for some bijection $\sigma:\{1,\ldots,L\}\to\{1,\ldots,L\}$. We adapt lemma 14 in Doukhan & Louhichi [?] to the series $(g_{t,s,w}(\eta,x))_{x\in B_n}$. For any integer $q\geq 1$, set :

$$A_q(n)=\sum_{\mathbf{x}\in B_n^q}|\mathbb{E}\Pi_{\mathbf{x},\mathbf{w}}|\,, \quad (23)$$

then,

$$\mathbb{E}(N_s^{B_n}-N_t^{B_n})^{2l}\leq |B_n|^{-l}A_{2l}(n). \quad (24)$$

If $q\geq 2$, for a multi-index $\mathbf{x}=(x_1,\ldots,x_q)$ of elements of S, the gap is defined by the maximum of the integers r such that the index may split into two non-empty sub-indices $\mathbf{y}=(y_1,\ldots,y_h)$ and $\mathbf{z}=(z_1,\ldots,z_{q-h})$ whose mutual distance equals r: $d(\mathbf{y}(\mathbf{x}),\mathbf{z}(\mathbf{x}))=\min\{d(y_a,z_b); 1\leq a\leq h, 1\leq b\leq q-h\}=r$. If the sequence is constant, its gap is 0.

Define the set $G_r(q,n) = \{\mathbf{x} \in B_n^q \text{ and the gap of } \mathbf{x} \text{ is } r\}$. Sorting the sequences of indices by their gap:

$$\begin{aligned} A_q(n) &\leq \sum_{x_1 \in B_n} \mathbb{E}|g_{s,t,w}(\eta, x_1)|^q + \sum_{r=1}^{n} \sum_{\mathbf{x} \in G_r(q,n)} \left|\mathrm{Cov}\left(\Pi_{\mathbf{y}(\mathbf{x}),\mathbf{w}}, \Pi_{\mathbf{z}(\mathbf{x}),\mathbf{w}}\right)\right| \qquad (25) \\ &+ \sum_{r=1}^{n} \sum_{\mathbf{x} \in G_r(q,n)} \left|\mathbb{E}\left(\Pi_{\mathbf{y}(\mathbf{x}),\mathbf{w}}\right) \mathbb{E}\left(\Pi_{\mathbf{z}(\mathbf{x}),\mathbf{w}}\right)\right|. \qquad (26) \end{aligned}$$

Denote

$$V_q(n) = \sum_{x_1 \in B_n} \mathbb{E}|g_{s,t,w}(\eta, x_1)|^q + \sum_{r=1}^{n} \sum_{\mathbf{x} \in G_r(q,n)} \left|\mathrm{Cov}\left(\Pi_{\mathbf{y}(\mathbf{x}),\mathbf{w}}, \Pi_{\mathbf{z}(\mathbf{x}),\mathbf{w}}\right)\right|.$$

In order to prove that the expression (26) is bounded by the product $\sum_h A_h(n) A_{q-h}(n)$ we make a first summation over the $\mathbf{x}$'s such that $\mathbf{y}(\mathbf{x}) \in B_n^h$. Hence:

$$A_q(n) \leq V_q(n) + \sum_{h=1}^{q-1} A_h(n) A_{q-h}(n).$$

To build a multi-index $\mathbf{x} = (x_1, \ldots, x_q)$ belonging to $G_r(q,n)$, we first fix one of the $|B_n|$ points of B_n, say x_1. We choose a second point x_2 with $d(x_1, x_2) = r$. The third point x_3 is in one of the ball with radius r centered in one of the previous points, and so on... Thus, because the maximal cardinality of a ball with radius r writes $b(r) \leq e^{\rho r}$

$$|G_r(q,n)| \leq |B_n| b(r) 2b(r) \cdots (q-1) b(r) \leq |B_n| (q-1)! 2^{q-1} e^{\rho(q-1)r}.$$

We use lemma 8.6 to deduce:

$$V_q(n) \leq |B_n| \left(K_\Omega |t-s| + (q-1)! 2^{q-1} \sum_{r=1}^{\infty} e^{\rho(q-1)r} (c_0 q e^{-cr} \wedge K_\Omega |t-s|) \right).$$

Let R be an integer to be specified, then

$$V_q(n) \leq |B_n| q! 2^{q-1} \left(K_\Omega |t-s| \sum_{r=0}^{R-1} e^{\rho(q-1)r} + c_0 \sum_{r=R}^{\infty} e^{(\rho(q-1)-c)r} \right).$$

Comparing those summations with integrals:

$$\begin{aligned} V_q(n) &\leq |B_n| q! 2^{q-1} \left(\frac{K_\Omega |t-s|}{\rho(q-1)} e^{\rho(q-1)R} + \frac{c_0}{c - \rho(q-1)} e^{(\rho(q-1)-c)(R-1)} \right) \\ &\leq |B_n| q! 2^{q-1} \frac{K_\Omega |t-s|}{c_1} e^{\rho(q-1)R} \left(1 + \frac{c_0}{K_\Omega |t-s|} e^{c-cR} \right), \end{aligned}$$

where $c_1 = \rho \wedge (c - \rho(2l-1))$. Assume that $(s,t) \in T$ are such that $|t-s| < c_0 e^c / K_\Omega$ and choose $R \geq 1$ as the integer such that $e^{c(R-1)} \leq \frac{c_0 e^c}{K_\Omega |t-s|} \leq e^{cR}$.

$$V_q(n) \leq |B_n| q! \frac{2^q K_\Omega |t-s| e^{2\rho(q-1)}}{c_1} \left(\frac{c_0}{K_\Omega |t-s|} \right)^{\frac{\rho(q-1)}{c}}, \tag{27}$$

so that $V_q(n)$ is a function of q that satisfies condition $(\mathcal{H}_0)$ of Doukhan & Louhichi [?]. Then

$$\begin{aligned} A_{2l}(n) &\leq \frac{(4l-2)!}{(2l)!(2l-1)!} \left(V_{2l}(n) + V_2(n)^l \right) \\ &\leq \frac{(4l-2)!(c_0 e^{2c})^{\frac{\rho l}{c}}}{(2l)!(2l-1)!} \left(\frac{2^{2l}(2l)!(c_0 e^{2c})^{\frac{\rho(l-1)}{c}}}{c_1} |B_n| (K_\Omega |t-s|)^{1-\frac{\rho(2l-1)}{c}} \right. \\ &\quad \left. + \left(\frac{8}{c_1} \right)^l |B_n|^l (K_\Omega |t-s|)^{l - \frac{\rho l}{c}} \right), \end{aligned}$$

and Proposition 8.7 is proved. □

To prove the tightness of the sequence of processes N^{B_n}, we study its oscillations:

$$w(\delta, N^{B_n}) = \sup_{\|t-s\|_1 < \delta} |N_t^{B_n} - N_s^{B_n}|$$

Fix ε and η. We have to find δ and n_0 such that for all $n > n_0$:

$$\mathbb{P}(w(\delta, N^{B_n}) \geq \varepsilon) \leq \eta$$

Define n_0 as the smallest integer such that $|B_{n_0}| > \delta^{-1-\rho/c}$, then for $n > n_0$, $|t-s| < \delta$, $l = 2$ and $c > 3\rho$, Proposition 8.7 yields:

$$\mathbb{E}(N_t^{B_n} - N_s^{B_n})^4 \leq C \delta^{2(1-\frac{\rho}{c})}$$

and we now follow the proof in Billingsley [?] to conclude.

8.3 Proof of Theorem 6.1

The proof is close to that of the analogous result in [?]. The convergence in distribution of $Z_n = (Z_n(t))_{t \geq 0}$, where $Z_n(t) = (D_t^{(n)} - |B_n| \cdot m(t))/\sqrt{|B_n|}$, does not directly imply the CLT for T_n. The Skorohod-Dudley-Wichura representation theorem is a much stronger result (see Pollard [?], section IV.3). It implies that there exist versions Z_n^* of Z_n and non-decreasing functions ϕ_n such that for any fixed s such that for Z^*, limit in distribution of Z_n:

$$\lim_{n \to \infty} \sup_{0 \leq t \leq s} |Z_n^*(t) - Z^*(\phi_n(t))| = 0 \quad a.s.$$

and:

$$\lim_{n\to\infty} \sup_{0\le t\le s} |\phi_n(t) - t| = 0 \quad a.s.$$

Since Z^* has continuous paths, it is uniformly continuous on $[0, s]$, and hence:

$$\lim_{n\to\infty} \sup_{0\le t\le s} |Z_n^*(t) - Z^*(t)| = 0 \quad a.s. \ , \tag{28}$$

We shall first use (28) to prove that the distributions of $\sqrt{|B_n|}(T_n - t_\alpha)$ are a tight sequence. Let c be a positive constant. On the one hand, if $D^{(n)}_{t_\alpha + c/\sqrt{|B_n|}} \ge k(n)$, then $T_n \le t_\alpha + c/\sqrt{|B_n|}$. Thus:

$$\begin{aligned}
\mathbb{P}[\sqrt{|B_n|}(T_n - t_\alpha) \le c] &\ge \mathbb{P}[D^{(n)}_{t_\alpha + c/\sqrt{|B_n|}} \ge k(n)] \\
&= \mathbb{P}[Z_n^*(t_\alpha + c/\sqrt{|B_n|}) \ge \sqrt{|B_n|}(\alpha - m(t_\alpha + c/\sqrt{|B_n|})) + o(1)] \\
&= \mathbb{P}[Z_n^*(t_\alpha + c/\sqrt{|B_n|}) \ge -cm'(t_\alpha) + o(1)] \\
&= \mathbb{P}[Z^*(t_\alpha) \ge -cm'(t_\alpha)] + o(1) \ ,
\end{aligned}$$

using (28) and the continuity of Z^*. Since $m'(t_\alpha) > 0$, we obtain that:

$$\lim_{c\to\infty} \liminf_{n\to\infty} \mathbb{P}[\sqrt{|B_n|}(T_n - t_\alpha) \le c] = 1. \tag{29}$$

On the other hand, we have:

$$\mathbb{P}[\sqrt{|B_n|}(T_n - t_\alpha) \le -c] = \mathbb{P}[\exists t \le t_\alpha - c/\sqrt{|B_n|} \ , \ Z_n^*(t) \ge \sqrt{|B_n|}(\alpha - m(t)) + o(1)].$$

But since the function m is increasing, for all $t \le t_\alpha - c/\sqrt{|B_n|}$ we have:

$$\sqrt{|B_n|}(\alpha - m(t)) \ge \sqrt{|B_n|}(\alpha - m(t_\alpha - c/\sqrt{|B_n|})) = cm'(t_\alpha) + o(1).$$

Hence:

$$\begin{aligned}
\mathbb{P}[\sqrt{|B_n|}(T_n - t_\alpha) \le -c] &\le \mathbb{P}[\exists t \le t_\alpha - c/\sqrt{|B_n|} \, , Z_n^*(t) \ge cm'(t_\alpha) + o(1)] \\
&\le \mathbb{P}[\exists t \le t_\alpha \, , Z_n^*(t) \ge cm'(t_\alpha) + o(1)] \\
&= \mathbb{P}[\exists t \le t_\alpha \, , Z^*(t) \ge cm'(t_\alpha) + o(1)] + o(1).
\end{aligned}$$

The process Z being *a.s.* bounded on any compact set and $m'(t)$ being positive on $[0, \tau]$, we deduce that:

$$\lim_{c\to\infty} \limsup_{n\to\infty} \mathbb{P}[\sqrt{|B_n|}(T_n - t_\alpha) \le -c] = 0. \tag{30}$$

Now (29) and (30) mean that the sequence of distributions of $(\sqrt{|B_n|}(T_n - t_\alpha))$ is tight. Hence to conclude it is enough to check the limit.

Using again (28), together with the almost sure continuity of Z yields:

$$\begin{aligned}
D^{(n)}_{t_\alpha+c/\sqrt{|B_n|}} &= |B_n|m(t_\alpha+u/\sqrt{|B_n|})+\sqrt{|B_n|}Z^*(t_\alpha+u/\sqrt{|B_n|})+o(\sqrt{|B_n|}) \quad a.s. \\
&= |B_n|\alpha+u\sqrt{|B_n|}m'(t_\alpha)+\sqrt{|B_n|}Z^*(t_\alpha)+o(\sqrt{|B_n|}) \quad a.s.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\inf\left\{u\,;D^{(n)}_{t_\alpha+u/\sqrt{|B_n|}}\geq k(n)\right\} &= \inf\left\{u\,;u\sqrt{|B_n|}m'(t_\alpha)+\sqrt{|B_n|}Z^*(t_\alpha)+o(\sqrt{|B_n|})\geq 0\right\} \\
&= -\frac{Z^*(t_\alpha)}{m'(t_\alpha)}+o(1).
\end{aligned}$$

The distribution of $-Z^*(t_\alpha)/m'(t_\alpha)$ is normal with mean 0 and variance σ_α^2, hence the result.

8.4 Proof of Proposition 7.1

Let $\mathcal{F}_{2,3}$ be the set of real valued functions h defined on $\mathbb{R}$, three times differentiable, such that $h(0)=0$, $\|h''\|_\infty<+\infty$, and $\|h^{(3)}\|_\infty<+\infty$. For a function $h\in\mathcal{F}_{2,3}$, we will denote by b_2 and b_3 the supremum norm of its second and third derivatives. We first need the following lemma.

Lemma 8.8 *Let h be a fixed function of the set $\mathcal{F}_{2,3}$. Let R be a fixed and finite subset of S. Let r be a fixed positive real. For any $x\in R$, let $V_x=B(x,r)\cap R$. Let $(Y_x)_{x\in S}$ be a real valued random field. Suppose that, for any $x\in S$, $\mathbb{E}Y_x=0$ and $\mathbb{E}Y_x^2<+\infty$. Let $Z(R)=\sum_{x\in R}Y_x$. Then*

$$\begin{aligned}
&\left|\mathbb{E}(h(Z(R)))-\operatorname{Var}Z(R)\int_0^1 t\mathbb{E}(h''(tZ(R)))dt\right| \\
&\leq \int_0^1\sum_{x\in R}\left|\operatorname{Cov}\left(Y_x,h'(tZ(V_x^c))\right)\right|dt+2\sum_{x\in R}\mathbb{E}|Y_x||Z(V_x)|\left[b_2\wedge b_3|Z(V_x)|\right] \\
&+b_2\mathbb{E}\left|\sum_{x\in R}\left(Y_xZ(V_x)-\mathbb{E}(Y_xZ(V_x))\right)\right|+b_2\sum_{x\in R}\left|\operatorname{Cov}(Y_x,Z(V_x^c))\right|, \qquad (31)
\end{aligned}$$

where $V_x^c=R\setminus V_x$.

Remark. For an independent random field $(Y_x)_{x\in S}$, fulfilling $\sup_{x\in S}\mathbb{E}Y_x^4<+\infty$, Lemma 8.8 applied with $V_x=\{x\}$, ensures

$$\left|\mathbb{E}(h(Z(R)))-\operatorname{Var}Z(R)\int_0^1 t\mathbb{E}(h''(tZ(R)))dt\right|\leq 2\sum_{x\in R}\mathbb{E}|Y_x|^2\left(b_2\wedge b_3|Y_x|\right)+b_2\sqrt{|R|}\sup_{x\in S}\|Y_x^2\|_2.$$

Proof of Lemma 8.8. We have,

$$
\begin{aligned}
h(Z(R)) &= Z(R)\int_0^1 h'(tZ(R))dt = \int_0^1 \left(\sum_{x\in R} Y_x h'(tZ(R))\right) dt \\
&= \int_0^1 \left(\sum_{x\in R} Y_x h'(tZ(V_x^c))\right) dt + \int_0^1 \left(\sum_{x\in R} Y_x \left(h'(tZ(R)) - h'(tZ(V_x^c)) - tZ(V_x)h''(tZ(R))\right)\right) dt \\
&+ \sum_{x\in R} Y_x Z(V_x) \int_0^1 th''(tZ(R))dt - \sum_{x\in R} \mathbb{E}\left(Y_x Z(V_x)\right) \int_0^1 th''(tZ(R))dt \\
&+ \sum_{x\in R} \mathbb{E}\left(Y_x Z(V_x)\right) \int_0^1 th''(tZ(R))dt - \sum_{x\in R} \mathbb{E}\left(Y_x Z(R)\right) \int_0^1 th''(tZ(R))dt \\
&+ \sum_{x\in R} \mathbb{E}\left(Y_x Z(R)\right) \int_0^1 th''(tZ(R))dt.
\end{aligned}
$$

We take expectation in the last equality. The obtained formula, together with the following estimations, proves Lemma 8.8.

$$
\begin{aligned}
&|h'(tZ(R)) - h'(tZ(V_x^c)) - tZ(V_x)h''(tZ(R))| \\
&\quad \le |h'(tZ(R)) - h'(tZ(V_x^c)) - tZ(V_x)h''(tZ(V_x^c))| + |Z(V_x)||h''(tZ(R)) - h''(tZ(V_x^c))| \\
&\quad \le 2|Z(V_x)|\left(b_2 \wedge b_3|Z(V_x)|\right). \qquad \square
\end{aligned}
$$

Our purpose now is to control the right hand side of the bound (31) for a random field $(Y_x)_{x\in S}$ fulfilling the covariance inequality (9) and the requirements of Proposition 7.1.

Corollary 8.9 *Let h be a fixed function of the set $\mathcal{F}_{2,3}$. Let R be a finite subset of S. For any $x \in R$ and for any positive real r, let $V_x = B(x,r)\cap R$. Let $(Y_x)_{x\in S}$ be a real valued random field, fulfilling the covariance inequality (9). Suppose that, for any $x \in S$, $\mathbb{E}Y_x = 0$ and $\sup_{x\in S}\|Y_x\|_\infty < M$, for some positive real M. Recall that $Z(R) = \sum_{x\in R} Y_x$. Then, for any $\delta > 0$, there exists a positive constant $C(\delta, M)$ independent of R, such that*

$$
\begin{aligned}
&\sup_{h\in\mathcal{F}_{2,3}} \left|\mathbb{E}(h(Z(R))) - \operatorname{Var} Z(R)\int_0^1 t\mathbb{E}(h''(tZ(R)))dt\right| \\
&\quad \le C(\delta, M)\left\{ b_2|R|e^{-\delta r} + b_3|R|\kappa_r + b_2|R|^{1/2}\kappa_r\left(\sum_{k=[3r]}^{\infty}\kappa_k e^{-\delta(k-2r)}\right)^{1/2}\right. \\
&\quad \left. + b_2|R|^{1/2}\kappa_{3r}\left(\sum_{k=1}^{[3r]+1} e^{-\delta k}\kappa_k\right)^{1/2}\right\},
\end{aligned}
$$

recall that $\sup_{x\in S}|B(x,n)| \le \kappa_n$.

Proof of Corollary 8.9
We have
$$V_x^c = \{y \in S, \quad d(x,y) \geq r\} \cap R.$$
Hence
$$d(\{x\}, V_x^c) \geq r.$$
The last bound together with (9), proves that
$$\begin{aligned}\sum_{x\in R} \left|\mathrm{Cov}\left(Y_x, h'(tZ(V_x^c))\right)\right| &\leq C_\delta b_2 \sum_{x \in R} (|V_x^c| \wedge 1) e^{-\delta d(\{x\}, V_x^c)} \\ &\leq C_\delta b_2 |R| e^{-\delta r}. \qquad (32)\end{aligned}$$
In the same way, we prove that
$$b_2 \sum_{x \in R} |\mathrm{Cov}(Y_x, Z(V_x^c))| \leq C_\delta b_2 |R| e^{-\delta r}. \qquad (33)$$
Now
$$\begin{aligned}\sum_{x \in R} \mathbb{E}|Y_x||Z(V_x)|\left(b_2 \wedge b_3|Z(V_x)|\right) &\leq b_3 M |R| \sup_{x\in S} \mathbb{E}|Z(V_x)|^2 \\ &\leq b_3 M |R| \kappa_r \sup_{y \in S} \sum_{z \in S} |\mathrm{Cov}(Y_y, Y_z)| \qquad (34)\end{aligned}$$
The last bound is obtained since $|V_x| \leq \kappa_r$ and $\sup_{y\in S}\sum_{z\in S}|\mathrm{Cov}(Y_y, Y_z)| < \infty$ (the proof of the last inequality is done along the same lines as that of Proposition 7.2) .
It remains to control
$$\mathbb{E}\left|\sum_{x\in R}\left(Y_x Z(V_x) - \mathbb{E}(Y_x Z(V_x))\right)\right|.$$
For this, we argue as Bolthausen [**?**]. We have
$$\begin{aligned}\mathbb{E}\left|\sum_{x\in R}\left(Y_x Z(V_x) - \mathbb{E}(Y_x Z(V_x))\right)\right|^2 &= \mathrm{Var}\left(\sum_{x\in R} Y_x Z(V_x)\right) \\ &= \sum_{x\in R}\sum_{y\in R} \mathrm{Cov}(Y_x Z(V_x), Y_y Z(V_y)).\end{aligned}$$
Hence, since $V_x \subset B(x,r)$,
$$\mathbb{E}\left|\sum_{x\in R}\left(Y_x Z(V_x) - \mathbb{E}(Y_x Z(V_x))\right)\right|^2 \leq \sum_{x\in R}\sum_{x'\in B(x,r)}\sum_{y\in R}\sum_{y'\in B(y,r)} \left|\mathrm{Cov}(Y_x Y_{x'}, Y_y Y_{y'})\right|. \qquad (35)$$
We have,
$$|\mathrm{Cov}(Y_x Y_{x'}, Y_y Y_{y'})| \leq |\mathrm{Cov}(Y_x Y_{x'}, Y_y Y_{y'})|\, \mathbb{I}_{d(x,y)\geq 3r} + |\mathrm{Cov}(Y_x Y_{x'}, Y_y Y_{y'})|\, \mathbb{I}_{d(x,y)\leq 3r}. \qquad (36)$$

We begin by controlling the first term. The covariance inequality (9) together with some elementary estimations, ensures

$$\begin{aligned}
|\mathrm{Cov}(Y_xY_{x'}, Y_yY_{y'})|\, \mathbb{I}_{d(x,y)\geq 3r} &\leq \sum_{k=[3r]}^{\infty} |\mathrm{Cov}(Y_xY_{x'}, Y_yY_{y'})|\, \mathbb{I}_{k\leq d(x,y)<k+1} \\
&\leq 2M^2 C_\delta \sum_{k=[3r]}^{\infty} e^{-\delta d(\{x,x'\},\{y,y'\})} \mathbb{I}_{k\leq d(x,y)<k+1} \\
&\leq 2M^2 C_\delta \sum_{k=[3r]}^{\infty} e^{-\delta(k-2r)} \mathbb{I}_{d(x,y)<k+1},
\end{aligned}$$

the last bound is obtained since, for any $x' \in B(x,r)$ and $y' \in B(y,r)$, we have,

$$d(\{x,x'\},\{y,y'\}) + 2r \geq d(\{x,x'\},\{y,y'\}) + d(x,x') + d(y,y') \geq d(x,y).$$

Hence,

$$\begin{aligned}
&\sum_{x\in R} \sum_{x'\in B(x,r)} \sum_{y\in R} \sum_{y'\in B(y,r)} |\mathrm{Cov}(Y_xY_{x'}, Y_yY_{y'})|\, \mathbb{I}_{d(x,y)\geq 3r} \\
&\leq 2M^2 C_\delta \kappa_r^2 \sum_{k=[3r]}^{\infty} \sum_{x\in R} \sum_{y\in R} e^{-\delta(k-2r)} \mathbb{I}_{y\in B(x,k+1)} \\
&\leq 2M^2 C_\delta |R| \kappa_r^2 \sum_{k=[3r]}^{\infty} \kappa_{k+1} e^{-\delta(k-2r)}.
\end{aligned} \tag{37}$$

We now control the second term in (36). Inequality (9) and the fact that $d(\{x\},\{x',y,y'\}) \leq d(\{x\},\{x'\})$, ensure

$$\begin{aligned}
&|\mathrm{Cov}(Y_xY_{x'}, Y_yY_{y'})|\, \mathbb{I}_{d(x,y)\leq 3r} \\
&\quad\leq |\mathrm{Cov}(Y_x, Y_{x'}Y_yY_{y'})|\, \mathbb{I}_{d(x,y)\leq 3r} + |\mathrm{Cov}(Y_x, Y_{x'})|\, |\mathrm{Cov}(Y_y, Y_{y'})|\, \mathbb{I}_{d(x,y)\leq 3r} \\
&\quad\leq 2M^2 C_\delta e^{-\delta d(\{x\},\{x',y,y'\})} \mathbb{I}_{d(x,y)\leq 3r}.
\end{aligned}$$

We deduce, using the last bound, that

$$\begin{aligned}
&|\mathrm{Cov}(Y_xY_{x'}, Y_yY_{y'})|\, \mathbb{I}_{d(x,y)\leq 3r} \\
&\leq \sum_{k=1}^{[3r]+1} |\mathrm{Cov}(Y_xY_{x'}, Y_yY_{y'})|\, \mathbb{I}_{d(x,y)\leq 3r} \mathbb{I}_{k-1\leq d(\{x\},\{x',y,y'\})<k} \\
&\leq 2M^2 C_\delta \sum_{k=1}^{[3r]+1} e^{-\delta(k-1)} \mathbb{I}_{d(x,y)\leq 3r} \mathbb{I}_{d(\{x\},\{x',y,y'\})<k}.
\end{aligned} \tag{38}$$

We have

$$\mathbb{I}_{d(\{x\},\{x',y,y'\})\leq k} \leq \mathbb{I}_{d(\{x\},\{x'\})\leq k} + \mathbb{I}_{d(\{x\},\{y\})\leq k} + \mathbb{I}_{d(\{x\},\{y'\})\leq k}.$$

Hence, we check that,

$$\sum_{x\in R}\sum_{x'\in B(x,r)}\sum_{y\in R}\sum_{y'\in B(y,r)}\mathbb{I}_{d(x,y)\le 3r}\mathbb{I}_{d(\{x\},\{x',y,y'\})\le k}\le 3|R|\kappa_{3r}^2\kappa_k. \tag{39}$$

We obtain combining (38) and (39),

$$\sum_{x\in R}\sum_{x'\in B(x,r)}\sum_{y\in R}\sum_{y'\in B(y,r)}|\mathrm{Cov}(Y_xY_{x'},Y_yY_{y'})|\,\mathbb{I}_{d(x,y)\le 3r}$$
$$\le 6e^{\delta}M^2C_\delta|R|\kappa_{3r}^2\sum_{k=1}^{[3r]+1}e^{-\delta k}\kappa_k. \tag{40}$$

We collect the bounds (35), (37) and (40), we obtain,

$$\mathbb{E}\left|\sum_{x\in R}(Y_xZ(V_x)-\mathbb{E}(Y_xZ(V_x)))\right|$$
$$\le C(\delta,M)|R|^{1/2}\left\{\kappa_r\left(\sum_{k=[3r]}^{\infty}\kappa_{k+1}e^{-\delta(k-2r)}\right)^{1/2}+\kappa_{3r}\left(\sum_{k=1}^{[3r]+1}e^{-\delta k}\kappa_k\right)^{1/2}\right\}. \tag{41}$$

Finally, the bounds (32), (33), (34), (41), together with Lemma 8.8 prove Corollary 8.9. □

End of the proof of Proposition 7.1. We apply Corollary 8.9 to the real and imaginary parts of the function $x\to\exp(iux/\sqrt{|B_n|})-1$. Those functions belong to the set $\mathcal{F}_{2,3}$, with $b_2=\dfrac{u^2}{|B_n|}$ and $b_3=\dfrac{|u|^3}{|B_n|^{3/2}}$.
We obtain, noting by ϕ_n the characteristic function of the normalized sum $Z(B_n)/\sqrt{|B_n|}$,

$$\left|\phi_n(u)-1+\frac{\mathrm{Var}\,Z(B_n)}{|B_n|}u^2\int_0^1 t\phi_n(tu)dt\right|$$
$$\le C(\delta,M,u)\left\{e^{-\delta r}+\frac{\kappa_r}{\sqrt{|B_n|}}+\frac{\kappa_r}{\sqrt{|B_n|}}\left(\sum_{k=[3r]}^{\infty}\kappa_ke^{-\delta(k-2r)}\right)^{1/2}\right.$$
$$\left.+\frac{\kappa_{3r}}{\sqrt{|B_n|}}\left(\sum_{k=1}^{[3r]+1}e^{-\delta k}\kappa_k\right)^{1/2}\right\}.$$

Let δ be a fixed positive real such that $\delta>12\rho$, recall that

$$\sup_{x\in S}|B(x,r)|\le 2e^{r\rho}=:\kappa_r.$$

Hence

$$\left|\phi_n(u) - 1 + \frac{\operatorname{Var} Z(B_n)}{|B_n|} u^2 \int_0^1 t\phi_n(tu)dt\right|$$

$$\leq C(\delta, M, u) \left\{ e^{-\delta r} + \frac{e^{r\rho}}{\sqrt{|B_n|}} + \frac{e^{(\rho+\delta)r}}{\sqrt{|B_n|}} \left(\sum_{k=[3r]}^{\infty} e^{-(\delta-\rho)k} \right)^{1/2} + \frac{e^{3\rho r}}{\sqrt{|B_n|}} \left(\sum_{k=1}^{[3r]+1} e^{-(\delta-\rho)k} \right)^{1/2} \right\}$$

$$\leq C(M, \rho, \delta, u) \left(e^{-\delta r} + \frac{e^{3r\rho}}{\sqrt{|B_n|}} + \frac{e^{-(\delta-5\rho)r/2}}{\sqrt{|B_n|}} \right).$$

For a suitable choice of the sequence r (for example we can take $r = \frac{2}{\delta} \ln |B_n|$), the right hand side of the last bound tends to 0 an n tends to infinity:

$$\lim_{n\to\infty} \left|\phi_n(u) - 1 + \frac{\operatorname{Var} Z(B_n)}{|B_n|} u^2 \int_0^1 t\phi_n(tu)dt\right| = 0. \tag{42}$$

We now need the following lemma.

Lemma 8.10 *Let σ^2 be a positive real. Let (X_n) be a sequence of real valued random variables such that $\sup_{n\in\mathbb{N}} \mathbb{E}X_n^2 < +\infty$. Let ϕ_n be the characteristic function of X_n. Suppose that for any $u \in \mathbb{R}$,*

$$\lim_{n\to+\infty} \left|\phi_n(u) - 1 + \sigma^2 \int_0^u t\phi_n(t)dt\right| = 0. \tag{43}$$

Then, for any $u \in \mathbb{R}$,

$$\lim_{n\to+\infty} \phi_n(u) = \exp(-\frac{u^2\sigma^2}{2}).$$

Proof of Lemma 8.10. Lemma 8.10 is a variant of Lemma 2 in Bolthausen [?]. The Markov inequality and the condition $\sup_{n\in\mathbb{N}} \mathbb{E}X_n^2 < +\infty$ imply that the sequence $(\mu_n)_{n\in\mathbb{N}}$ of the laws of (X_n) is tight. Theorem 25.10 in Billingsley [?] proves the existence of a subsequence μ_{n_k} and a probability measure μ such that μ_{n_k} converges weakly to μ as k tends to infinity. Let ϕ be the characteristic function of μ. We deduce from (43) that, for any $u \in \mathbb{R}$,

$$\phi(u) - 1 + \sigma^2 \int_0^u t\phi(t)dt = 0,$$

or equivalently, for any $u \in \mathbb{R}$,

$$\phi'(u) + \sigma^2 u\phi(u) = 0.$$

We obtain, integrating the last equation, that for any $u \in \mathbb{R}$,

$$\phi(u) = \exp(-\frac{\sigma^2u^2}{2}).$$

The proof of Lemma 8.10 is completed by using Theorem 25.10 in Billingsley [?] and its corollary. $\square$

Proposition 7.1 follows from (11), (42) and Lemma 8.10. $\square$

8.5 Proof of Proposition 7.2.

We deduce from (9) that for any positive real δ there exists a positive constant C_δ such that for different sites x and y of S,

$$|\mathrm{Cov}(Y_x, Y_y)| \le C_\delta e^{-\delta d(x,y)}. \tag{44}$$

Hence, the first conclusion of Proposition 7.2 follows from the bound (44), together with the following elementary calculations, for $\rho < \delta$,

$$\begin{aligned} \sum_{z\in S} |\mathrm{Cov}(Y_0, Y_z)| &\le C_\delta \sum_{z\in S} \exp(-\delta d(0,z)) \\ &\le C_\delta \sum_{z\in S} \sum_{r=0}^{\infty} \exp(-\delta d(0,z)) \mathbb{I}_{r\le d(0,z)<r+1} \\ &\le C_\delta \sum_{r=0}^{\infty} \exp(-\delta r) \sum_{z\in S} \mathbb{I}_{d(0,z)<r+1} \\ &\le C_\delta \sum_{r=0}^{\infty} \exp(-\delta r)|B(0, r+1)| \\ &\le C(\delta,\rho) \sum_{r=0}^{\infty} \exp(-(\delta-\rho)r), \end{aligned} \tag{45}$$

where $C(\delta, \rho)$ is a positive constant depending on δ and ρ.
We now prove the second part of Proposition 7.2. Thanks to (7), we can find a sequence $u = (u_n)$ of positive real numbers such that

$$\lim_{n\to+\infty} u_n = +\infty, \lim_{n\to+\infty} \frac{|\partial B_n|}{|B_n|} \exp(\rho u_n) = 0. \tag{46}$$

Let $(\partial_u B_n)_n$ be the sequence of subsets of S defined by

$$\partial_u B_n = \{s \in B_n \,:\, d(s, \partial B_n) < u_n\}.$$

The bound (4) gives

$$|\partial_u B_n| \le 2|\partial B_n| e^{u_n \rho},$$

which together with the suitable choice of the sequence (u_n) ensures

$$\lim_{n\to+\infty} \frac{|\partial_u B_n|}{|B_n|} = 0, \tag{47}$$

we shall use this fact below without further comments. Let $B_n^u = B_n \backslash \partial_u B_n$. We decompose the quantity Var S_n as in Newman [**?**]:

$$\frac{1}{|B_n|}\mathrm{Var}\ S_n = \frac{1}{|B_n|} \sum_{x\in B_n} \sum_{y\in B_n} \mathrm{Cov}\,(Y_x, Y_y) = T_{1,n} + T_{2,n} + T_{3,n},$$

where

$$\begin{aligned} T_{1,n} &= \frac{1}{|B_n|}\sum_{x\in B_n^u}\sum_{y\in B_n\setminus B(x,u_n)} \mathrm{Cov}\left(Y_x, Y_y\right), \\ T_{2,n} &= \frac{1}{|B_n|}\sum_{x\in B_n^u}\sum_{y\in B_n\cap B(x,u_n)} \mathrm{Cov}\left(Y_x, Y_y\right), \\ T_{3,n} &= \frac{1}{|B_n|}\sum_{x\in \partial_u B_n}\sum_{y\in B_n} \mathrm{Cov}\left(Y_x, Y_y\right). \end{aligned}$$

Control of $T_{1,n}$. We have, since $|B_n^u| \leq |B_n|$ and applying (44)

$$|T_{1,n}| \leq \sup_{x\in S}\sum_{y\in S\setminus B(x,u_n)} |\mathrm{Cov}(Y_x, Y_y)| \leq C_\delta \sup_{x\in S}\sum_{y\in S\setminus B(x,n)} \exp(-\delta d(x,y)). \tag{48}$$

For any fixed $x \in S$, we argue as for (45) and we obtain for $\rho < \delta$,

$$\sum_{y\in S\setminus B(x,n)} \exp(-\delta d(x,y)) \leq C(\delta)\sum_{r=[u_n]}^{\infty} \exp(-(\delta-\rho)r) \leq C(\delta,\rho)\exp(-(\delta-\rho)u_n) \tag{49}$$

We obtain, collecting (48), (49) together with the first limit in (46) :

$$\lim_{n\to+\infty} T_{1,n} = 0. \tag{50}$$

Control of $T_{3,n}$. We obtain using (44) :

$$|T_{3,n}| \leq \frac{|\partial_u B_n|}{|B_n|}\sup_{x\in S}\sum_{y\in S} |\mathrm{Cov}(Y_x, Y_y)|. \tag{51}$$

The last bound, together with the limit (47) gives

$$\lim_{n\to+\infty} T_{3,n} = 0. \tag{52}$$

Control of $T_{2,n}$. We deduce using the following implication, if $x \in B_n^u$ and y is not belonging to B_n then $d(x,y) \geq u_n$, that

$$T_{2,n} = \frac{1}{|B_n|}\sum_{x\in B_n^u}\sum_{y\in B(x,u_n)} \mathrm{Cov}(Y_x, Y_y)$$

We claim that,

$$\sum_{y\in B(x,u_n)} \mathrm{Cov}(Y_x, Y_y) = \sum_{z\in B(0,u_n)} \mathrm{Cov}\left(Y_0, Y_z\right), \tag{53}$$

in fact, since the graph $\mathcal{G}$ is transitive, there exits an automorphism a_x, such that $a_x(x) = 0$ (0 is a fixed vertex in S). Equality (10) gives

$$\sum_{y \in B(x,u_n)} \mathrm{Cov}(Y_x, Y_y) = \sum_{y \in B(x,u_n)} \mathrm{Cov}(Y_0, Y_{a_x(y)}).$$

Now, Lemma 1.3.2 in Godsil and Royle [**?**] yields that $d(x, y) = d(a_x(x), a_x(y)) = d(0, a_x(y))$. From this we deduce that $y \in B(x, u_n)$ if and only if $a_x(y) \in B(0, u_n)$. From above, we conclude that,

$$\sum_{y \in B(x,u_n)} \mathrm{Cov}(Y_x, Y_y) = \sum_{a_x(y) \in B(0,u_n)} \mathrm{Cov}(Y_0, Y_{a_x(y)}) = \sum_{z \in B(0,u_n)} \mathrm{Cov}(Y_0, Y_z),$$

which proves (53). Consequently,

$$T_{2,n} \quad = \quad \frac{|B_n^u|}{|B_n|} \sum_{z \in B(0,u_n)} \mathrm{Cov}(Y_0, Y_z).$$

The last equality together with the first limit in (46) and (47), ensures

$$\lim_{n \to +\infty} T_{2,n} = \sum_{z \in S} \mathrm{Cov}(Y_0, Y_z). \tag{54}$$

The second conclusion of Proposition 7.2 is proved by collecting the limits (50), (52) and (54). $\square$

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